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ROBUST STABILIZATION OF LINEAR INTERVAL SYSTEMS†

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For systems whose parameters are accurately known up to their upper and lower limits, a ranked controllability criterion is introduced using interval matrices [1–3]. A procedure is proposed for calculating the minimum singular number of the interval matrix, which serves as a measure of the controllability margin [4]. The controllability criterion introduced is used to synthesize robust control. It is shown that the parameters of the controller with the required properties can be found by solving the Sylvester equation with interval coefficients. © 2002 Elsevier Science Ltd. All rights reserved.

1. FORMULATION OF THE PROBLEM

Consider a linear object with parametric indeterminacy, the mathematical model of which is represented by a vector differential equation with interval coefficients

$$\dot{x}(t)=\tilde{A}x(t)+\tilde{B}u(t),\ \ x(t_0)=x_0;\ \ y(t)=\tilde{C}x(t),\ \ t\in[t_0,\infty) \tag{1.1}$$

where $x(x)\in R^n$ is the vector of phase states, $u(t)\in R^m$ is the vector of control actions, $y(t)\in R^l$ is the vector of output (measured) coordinates $(l\leqslant n)$, and $\tilde{A}\in IR^{n\times n}, \tilde{B}\in IR^{n\times m}$ and $\tilde{C}\in IR^{l\times n}$ are interval matrixes (IMs) with elements $\tilde{a}_{ij}=[\underline{a}_{ij},\bar{a}_{ij}]$, $\tilde{b}_{ij}=[\underline{b}_{ij},\bar{b}_{ij}]$ and $\tilde{c}_{ij}=[\underline{c}_{ij},\bar{c}_{ij}]$ belonging to an extended set of intervals $IR=\{[\underline{r},\bar{r}\,|\,\underline{r},\bar{r}\in R\}$, which contains both correct intervals $(\underline{r},\leqslant\bar{r})$ and incorrect intervals $(\underline{r},\geqslant\bar{r})$.

The interval linear system (1.1) is understood as a family of "point" objects

$$\dot{x}(t)=Ax(t)+Bu(t),\ \ x(t_0)=x_0,\ \ t\in[t_0,\infty) \tag{1.2}$$

with real coefficients $A\in R^{n\times n}$ and $B\in R^{n\times m}$ from the specified intervals $A\in[\underline{A},\bar{A}]$ and $B\in[\underline{B},\bar{B}]$:

$$\dot{x}(t)=\tilde{A}x(t)+\tilde{B}u(t)\overset{\text{def}}{\Leftrightarrow}\{\dot{x}(t)=Ax(t)+Bu(t)\,|\,\forall A\in\tilde{A},\forall B\in\tilde{B}\}$$

Of the different problems of robust control for interval dynamic objects, we will examine the problem of robust stabilization. The problem of robust stabilization, or robust arrangement, of the spectrum consists of selecting the feedback $\tilde{K}\in IR^{m\times n}$ such that the spectrum $\rho(\tilde{A}_C)$ of the matrix of the closed system $\tilde{A}_C=(\tilde{A}-\tilde{B}\tilde{K})\in IR^{n\times n}$ belongs to the prescribed spectrum $\rho(-\tilde{F})$, which is set by the spectrum of the reference matrix $\tilde{F}$ and is positioned in the left-hand half-plane $\{\operatorname{Re}\lambda<0\}$

$$\rho(\tilde{A}_C)\subseteq\rho(-\tilde{F})\ \ \text{when all}\ \ A\in\tilde{A},B\in\tilde{B} \tag{1.3}$$

$$\tilde{F}=\operatorname{diag}(\tilde{\mu}_i)_{i=1}^n\in IR^{n\times n},\ \ \tilde{\mu}_i=\left\lfloor\underline{\mu}_i,\bar{\mu}_i\right\rfloor$$

where $\tilde{F}$ is an IM with diagonal elements that are selected on the basis of the requirements concerning the lower and upper limits of the direct quality indices of the transients in the closed system. The inclusion (1.3) is understood in a component-by-component sense

$$\tilde{\lambda}_i(\tilde{A}_C)=[\underline{\lambda}_i(\tilde{A}_C),\ \bar{\lambda}_i(\tilde{A}_C)]\subseteq\tilde{\lambda}_i(-\tilde{F})=[\underline{\lambda}_i(-\tilde{F}),\ \bar{\lambda}_i(-\tilde{F})],\ \ \forall i=1,\dots,n$$

where $\bar{\lambda}_i\,(\bullet)$. The order relation $\subseteq$ in the extended interval arithmetic [5] was defined by the relation in $\leqslant R$

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$$[\underline{\lambda}_i(\tilde{A}_C),\ \overline{\lambda}_i(\tilde{A}_C)] \subseteq [\underline{\lambda}_i(-\tilde{F}),\ \overline{\lambda}_i(-\tilde{F})] \overset{\text{def}}{\Leftrightarrow} ((\underline{\lambda}_i(\tilde{A}_C) \geqslant \underline{\lambda}_i(-\tilde{F})) \,\&\, (\overline{\lambda}_i(\tilde{A}_C) \leqslant \overline{\lambda}_i(-\tilde{F})))$$

Synthesis of the controls ensuring the desired robust property (1.3) in system (1.1) reduces to solving the Sylvester equation with interval coefficients. The constructive use of interval matrix equations in the practice of synthesizing robust control depends on the presence of reliable and readily checkable conditions for their solvability, related to the properties of controllability and observability.

2. INTERVAL CRITERION OF CONTROLLABILITY

If, for an arbitrarily prescribed initial state $x(t_0) = x_0$ and final state $x(t_1) = x_1$, a control $u(t)$ exists which converts system (1.2) with the parameters $A \in \tilde{A}$ and $B \in \tilde{B}$ within finite time $t_1 - t_0$ from a state x_0 into a state x_1, then the interval system (1.1) (interval matrix pair $(\tilde{A}, \tilde{B})$) will be said to be controllable.

We will define the matrix of controllability for interval pair $(\tilde{A}, \tilde{B})$ as follows:

$$\tilde{D} = \left\| \tilde{B} \middle| \tilde{A}\tilde{B} \middle| \ldots \middle| \tilde{A}^{n-1}\tilde{B} \right\| \overset{\text{def}}{=} \{D \in R^{n\times mn},\ D = \left\| B \middle| AB \middle| \ldots \middle| A^{n-1}B \right\| \,|\, \forall A \in \tilde{A}, \forall B \in \tilde{B}\} \tag{2.1}$$

The product of the two IMs in expression (2.1) is calculated in a similar way to the multiplication of real matrices taking into account the fact that the product of the two intervals $\tilde{\upsilon}$ and $\tilde{u}$ in the complete interval arithmetic is defined by the following interval expression [6]

$$\tilde{\upsilon}\cdot\tilde{u} = [\max\{(\underline{\upsilon}^+\underline{u}^+),\ (\overline{\upsilon}^-\overline{u}^-)\} - \max\{(\overline{\upsilon}^+\underline{u}^-),\ (\underline{\upsilon}^-\overline{u}^+)\},$$

$$\max\{(\overline{\upsilon}^+\overline{u}^+),\ (\underline{\upsilon}^-\underline{u}^-)\} - \max\{(\underline{\upsilon}^+\overline{u}^-),\ (\overline{\upsilon}^-\underline{u}^+)\}]$$

where $\upsilon^+ = \max\{\upsilon, 0\}$ and $u^+ = \max\{u, 0\}$ are the positive parts and $\upsilon^- = \max\{-\upsilon, 0\}$ and $u^- = \max\{-u, 0\}$ are the negative parts of the real number υ and u.

We will assume that the rank of the IM $\tilde{H}$ is equal to n if an real matrix $H \in \tilde{H}$ has rank n:

$$\operatorname{rank}\tilde{H} = n \overset{\text{def}}{\Leftrightarrow} \{H \in R^{n\times mn} \mid \operatorname{rank} H = n,\ \ \forall H \in \tilde{H}\} \tag{2.2}$$

Then the interval analogue of the rank criterion of controllability can be formulated as follows.

Assertion 1. The interval system (1.1) (the interval pair $(\tilde{A}, \tilde{B})$) is controllable when, and only when,

$$\operatorname{rank}\tilde{D} = n \tag{2.3}$$

The correctness of Assertion 1 follows from the controllability of "point" systems with all possible values of their parameters and definition (2.2) of the rank of the IM.

Set (2.1) has the cardinality of a continuum. To check criterion (2.3), it is necessary to calculate the rank of the countless number of controllability matrices of points system (1.2).

3. CALCULATION OF THE RANK OF THE INTERVAL MATRIX

We will solve the problem of determining the rank of the controllability IM by factorizing it in the form of a singular expansion. The singular expansion of the real matrix $H \in R^{n\times m}$ of the rank k has the form

$$H = W\Sigma V \tag{3.1}$$

where $W \in R^{n\times n}$ and $V \in R^{m\times m}$ are orthogonal matrices whose columns are the left- and right-hand singular vectors of matrix H, identical with the eigenvectors of matrices HH^T and H^TH respectively. The matrix $\Sigma = (\sigma_{ij}) \in R^{n\times m}$ consists of a diagonal quadratic cell of dimension $q \times q$ ($q = \min\{m, n\}$) such that $\sigma_{ij} = 0$ when $i \neq j$, $\sigma_{11} \geqslant \sigma_{22} \geqslant \ldots \geqslant \sigma_{kk} > \sigma_{k+1,k+1} = \ldots \sigma_{qq} = 0$, and when $n \neq m$ of additional zero rows and columns. Diagonal elements $\sigma_i \equiv \sigma_{ii}$ of matrix Σ, termed singular numbers of matrix H, are non-negative square roots of the eigenvalues (EVs) of the matrix HH^T.

It is well known [7] that the rank of the arbitrary matrix H is equal to the rank of the diagonal cell of the matrix Σ in its singular expansion, and consequently the rank of the matrix can be defined as the number of non-zero singular numbers.

We will generalize the approach to determining the rank of the matrix by estimating the non-zero singular numbers for the interval case. For this, we will use the relation between the singular numbers of the matrices $H \in \tilde{H}$ and the EVs of the matrices $HH^T \in \tilde{H}\tilde{H}^T$.

We will apply as the singular expansion of IM $\tilde{H}$ a set of singular expansions of the real matrices $H \in \tilde{H}$

$$\tilde{H} = \tilde{W}\tilde{\Sigma}\tilde{V} \overset{\text{def}}{\Leftrightarrow} \{H = W\Sigma V \mid \forall H \in \tilde{H}\}$$

The matrices W, Σ and V were defined in the explanations to formula (3.1).

The set

$$\tilde{\sigma}_s(\tilde{H}) = [\underline{\sigma}_s(\tilde{H}), \overline{\sigma}_s(\tilde{H})] \overset{\text{def}}{\Leftrightarrow} \{\sigma_s(H), s = 1,\dots,q \quad q = \min\{n,m\} \mid \forall H \in \tilde{H}\} \tag{3.2}$$

consisting of singular numbers of all real matrices $H \in \tilde{H}$ will be termed the singular number of the IM $\tilde{H} \in IR^{n \times m}$.

Assertion 2. The singular numbers of the IM $\tilde{H} \in IR^{n \times m}$ are non-negative square roots of the EVs of the matrix $\tilde{H}\tilde{H}^T \in IR^{n \times n}$ or $\tilde{H}^T\tilde{H} \in IR^{m \times m}$:

$$\tilde{\sigma}_{q-j}(\tilde{H}) = \sqrt{\tilde{\lambda}_{m-j}(\tilde{H}^T\tilde{H})} = \sqrt{\tilde{\lambda}_{n-j}(\tilde{H}\tilde{H}^T)}, \; j = 1,\dots,q-1, \; q = \min\{n,m\} \tag{3.3}$$

Proof. Consider the matrix HH^T, the EVs $\lambda_i(HH^T)$ of which are numbered in order of increase $\lambda_1 \leqslant \lambda_2 \leqslant \dots \leqslant \lambda_n$. If w_i are the columns of the matrix W, then

$$HH^T w_i = \lambda_i(HH^T) w_i, \; (w_i, w_j) = \delta_{ij}; \; i,j = 1,\dots,n \tag{3.4}$$

where $(\cdot,\cdot)$ is a scalar product and δ_{ij} is the Kronecker delta.

Let the elements h_{ij} of the matrix $\tilde{H}$ vary within certain limits: $h_{ij} \in \tilde{h}_{ij} = [\underline{h}_{ij}, \bar{h}_{ij}]$. Then it is necessary to examine the set of matrices H that is described by the IM $\tilde{H} \in IR^{n \times m}$. When the elements of matrix H change, the elements of the matrix HH^T will also change. When the elements of the matrix H change, each representative $HH^T \in \tilde{H}\tilde{H}^T$, by virtue of the method of its construction, will remain symmetrical, which will keep the EVs real and preserve the orthogonality of the eigenvectors for each representative HH^T from set $\tilde{H}\tilde{H}^T$ in problem (3.4).

If $\hat{\lambda}_i(HH^T)$ is the approximate EV of the symmetrical matrix HH^T, and $\hat{w}_i$ is an approximate eigenvector normalized by the condition $\|\hat{w}_i\| = 1$, then the magnitude of the discrepancy of the left- and right-hand side of Eq. (3.4) when $\hat{\lambda}_i(HH^T)$ and $\hat{w}_i$ are substituted into it is equal to $\xi_i = HH^T\hat{w}_i - \hat{\lambda}_i(HH^T)\hat{w}_i$. The accurate EV $\lambda_i(HH^T)$ of the matrix HH^T will belong to the interval [8]

$$[\hat{\lambda}_i(HH^T) - \varepsilon_i, \; \hat{\lambda}_i(HH^T) + \varepsilon_i]$$

where $\varepsilon_i = \|\xi_i\|_2$ is the Euclidean norm of the discrepancy.

The singular numbers of the real matrix H and the EVs of the matrix HH^T or H^TH are connected by the relation [7]

$$\sigma_{q-j}(H) = \sqrt{\lambda_{m-j}(H^TH)} = \sqrt{\lambda_{n-j}(HH^T)}, \; j = 1,\dots,q-1, \; q = \min\{n,m\} \tag{3.5}$$

Changing to the IM, we establish that, for each symmetrical matrix $HH^T \in \tilde{H}\tilde{H}^T$, there is a set of eigenvalues lying in the range

$$\tilde{\lambda}_i(\tilde{H}\tilde{H}^T) = [\underline{\lambda}_i(\tilde{H}\tilde{H}^T), \; \overline{\lambda}_i(\tilde{H}\tilde{H}^T)] = [\lambda_i\{\mathrm{med}(\tilde{H}\tilde{H}^T)\} - \varepsilon_i, \; \lambda_i\{\mathrm{med}(\tilde{H}\tilde{H}^T)\} + \varepsilon_i] \tag{3.6}$$

where

$$\varepsilon_i = \left\|\tilde{\xi}_i\right\|_2, \tilde{\xi}_i = \tilde{H}\tilde{H}^T w_i - \lambda_i\{\mathrm{med}(\tilde{H}\tilde{H}^T)\} w_i \tag{3.7}$$

w_i is the eigenvector corresponding to the EV λ_i {med($\tilde{H}\tilde{H}^T$)}, med($\tilde{H}\tilde{H}^T$) = $\bar{H}\bar{H}^T + \underline{H}\underline{H}^T)/2$ is the median of the IM $\tilde{H}\tilde{H}^T$, $|\tilde{\xi}_i| = \mathrm{col}\left(|\tilde{\xi}_{ij}|\right)_{j=1}^{n}$ is the modulus of the interval vector $\tilde{\xi}_i$ and $|\tilde{\xi}_i| = \max\left\{|\underline{\xi}_{ij}|, |\bar{\xi}_{ij}|\right\}$ is the modulus of its jth component.

Let $Z \stackrel{\text{def}}{=} \{\tilde{z} \in IR \,|\, (\underline{z} > 0)\ \&\ (\bar{z} > 0)\}$ be a subset of non-negative intervals. We will define on the subset Z the operation of calculating the square root: $\tilde{a} \in Z$, $\sqrt{\tilde{a}} = \lfloor\sqrt{\underline{a}}, \sqrt{\bar{a}}\rfloor$. From the relation (3.5) and the definition of the singular number (3.2) there follows expression (3.3), in which the EVs of the IM $\tilde{H}\tilde{H}^T$ are calculated by means of formula (3.6).

Assertion 3. The rank of the IM $\tilde{H} \in IR^{n\times m}$ is equal to k when, and only when, the number of its singular numbers belonging to the subset of non-negative intervals is equal to k:

$$\mathrm{rank}\,\tilde{H} = k \stackrel{\text{def}}{\Leftrightarrow} \{\tilde{\sigma}_s(\tilde{H}) \in Z,\ \forall s = 1,\dots,k,\ k \leqslant q = \min\{n, m\}\}$$

In other words, for singular numbers of the IM, the following inequalities must be satisfied: $\underline{\sigma}_s(\tilde{D}) > 0$ and $\bar{\sigma}_s(\tilde{D}) > 0$. Assertion 3 follows from the properties of the singular expansion of the real matrix and the definition of the singular number of the IM.

Let $\langle\tilde{a}\rangle$ denote the least distance of points of the interval $\tilde{a}$ from zero [9]

$$\langle\tilde{a}\rangle \stackrel{\text{def}}{=} \begin{cases} \min\{|\underline{a}|, |\bar{a}|\}, & \text{if } 0 \notin \tilde{a} \\ 0, & \text{if } 0 \in \tilde{a} \end{cases}$$

The least distance from the minimum singular number to zero in the IR can serve to some degree as a measure of the remoteness of the controllability matrix $\tilde{D}$ from the set of degenerate matrices and, accordingly, can serve as a measure of the controllability margin.

$$\mu(\tilde{A}, \tilde{B}) = \langle\tilde{\sigma}_{\min}(\tilde{D})\rangle = \min\{|\underline{\sigma}_{\min}(\tilde{D})|, |\bar{\sigma}_{\min}(\tilde{D})|\} \tag{3.8}$$

4. THE PROBLEM OF THE ROBUST ARRANGEMENT OF THE SPECTRUM

The problem of the arrangement of the spectrum is one of the classical problems of the linear theory of optimum control. Russian scientists have made a considerable contribution to development of the theory of the analytical construction of controllers and methods of constructing stabilizing controls [10–13]. For systems with real coefficients, the problem of the arrangement of the spectrum or the problem of synthesizing a stabilizing controller, as is well known [14], can be reduced to the solution of Sylvester's equation. We will solve the problem of the robust arrangement of the spectrum of a dynamical system with interval indeterminacy of the parameters using the controllability criterion introduced. For this, we will examine the Sylvester interval matrix equation

$$\tilde{A}\tilde{P} + \tilde{P}\tilde{F} = \tilde{B}\tilde{G} \tag{4.1}$$

where $\tilde{G} \in IR^{m\times n}$ is an arbitrary matrix and $\tilde{F} \in IR^{n\times n}$ is the IM defining the desirable dynamics of the closed system and satisfying the requirement

$$\rho(\tilde{A}) \cap \rho(\tilde{F}) = \varnothing \tag{4.2}$$

where $\rho(\tilde{A}) = \{\rho(A) \,|\, \forall A \in \tilde{A},\ \rho(A) = \{\lambda_i(A),\ i = 1, \dots, n\}\}$ is the spectrum of the IM $\tilde{A}$, and $\rho(\tilde{F}) = \{\rho(F) \,|\, \forall F \in \tilde{F},\ \rho(F) = \{\lambda_i(F) = \mu_i \in \tilde{\mu}_i,\ i = 1, \dots, n\}\}$ is the spectrum of the reference matrix $\tilde{F}$ prescribed by the sequence $\{\tilde{\mu}_1, \dots, \tilde{\mu}_n\}$, in which the interval numbers $\tilde{\mu}_i = [\underline{\mu}_i, \bar{\mu}_i]$ are mutually different and $\tilde{\mu}_i > 0$. The intersection (4.2) is understood in the component-by-component sense

$$\tilde{\lambda}_i(\tilde{A}) \cap \tilde{\lambda}_i(\tilde{F}) = \varnothing, \quad \forall i = 1, \dots, n$$

for eigenvalues of the matrices $\tilde{A}$ and $\tilde{F}$ order according to increase (or decrease).

Interval control (4.1) is understood as a set of equations of similar structure

$$AP + PF = BG \tag{4.3}$$

in which the real coefficients $A \in R^{n\times n}$, $B \in R^{n\times m}$, $G \in R^{m\times n}$ and $F \in R^{n\times n}$ take all possible values from the prescribed range $\tilde{A}, \tilde{B}, \tilde{G}$ and $\tilde{F}$

$$\tilde{A}\tilde{P} + \tilde{P}\tilde{F} = \tilde{B}\tilde{G} \overset{\text{def}}{\Leftrightarrow} \{AP + PF = BG \mid \forall A \in \tilde{A},\ \forall B \in \tilde{B},\ \forall G \in \tilde{G},\ \forall F \in \tilde{F}\}$$

The set

$$\tilde{P} = \{P \in R^{n\times n} \mid (\forall A \in \tilde{A})(\forall B \in \tilde{B})(\forall G \in \tilde{G})(\forall F \in \tilde{F})(AP + PF = BG)\} \tag{4.4}$$

will be termed the combined set of solutions of the interval matrix equation (4.1).

The parameters $\tilde{K} \in IR^{m\times n}$ of the robust controller in the form of vector state feedback

$$u(x) = -\tilde{K}x \tag{4.5}$$

are found from the equation

$$\tilde{K}\tilde{P} = \tilde{G} \overset{\text{def}}{\Leftrightarrow} \{KP = G \mid \forall G \in \tilde{G},\ \forall P \in \tilde{P}\} \tag{4.6}$$

System (1.1), closed by controller (4.5) with the parameters

$$\tilde{K} = \{K \in R^{m\times n} \mid (\forall G \in \tilde{G})(\forall P \in \tilde{P})(KP = G)\} \tag{4.7}$$

takes the form

$$\dot{x}(t) = (\tilde{A} - \tilde{B}\tilde{K})x(t) = \tilde{A}_C x(t),\ \ x(t_0) = x_0 \tag{4.8}$$

and its properties are defined by the following theorem.

Theorem. Let
(1) the interval matrix pair $(\tilde{A}, \tilde{B})$ be controllable;
(2) the interval matrix pair $(\tilde{G}, \tilde{F})$ be observable;
(3) condition (4.2) be satisfied.

Then control (4.5) with parameters (4.7) ensures that the spectrum of the matrix $\tilde{A}_C$ of closed system (4.8) belongs to the spectrum of the reference matrix $(-\tilde{F})$ for all $A \in \tilde{A}$ and $B \in \tilde{B}$.

Remark. The set of robust controllers (4.5) with parameters (4.7) can be interpreted as a nominal controller with a permissible variation

$$\tilde{K} = K_0 \pm \delta K$$

where $K_0 = \operatorname{med}\tilde{K} = (\bar{K} + \underline{K})/2$ are nominal parameters, and $\delta K_0 = \operatorname{rad}\tilde{K} = (\bar{K} - \underline{K})/2$ is the tolerance on the nominal parameters.

The solution of Eq. (4.1) can be reduced to solving the system of interval algebraic equations [15]

$$\tilde{W}\tilde{p} = \tilde{e} \tag{4.9}$$

where

$$\tilde{W} = I_n \otimes \tilde{A} + \tilde{F}^T \otimes I_n = \begin{Vmatrix} \tilde{A} + \tilde{f}_{11}I_n & \tilde{f}_{21}I_n & \cdots & \tilde{f}_{n1}I_n \\ \tilde{f}_{12}I_n & \tilde{A} + \tilde{f}_{22}I_n & \cdots & \tilde{f}_{n2}I_n \\ \cdots & \cdots & \cdots & \cdots \\ \tilde{f}_{1n}I_n & \tilde{f}_{2n}I_n & \cdots & \tilde{A} + \tilde{f}_{nn}I_n \end{Vmatrix} \in IR^{n^2\times n^2}$$

$$\tilde{p} = \operatorname{col}(\tilde{P}) \in IR^{n^2},\quad \tilde{e} = \operatorname{col}(\tilde{E}) \in IR^{n^2},\quad I_n = \operatorname{diag}\{1\}_1^n,\quad \tilde{E} = \tilde{B}\tilde{G}$$

$\otimes$ is the direct or Kronecker product of the matrices, and $\operatorname{col}(\tilde{P}) = (\tilde{P}_{11}, \ldots, \tilde{P}_{n1}, \ldots, \tilde{P}_{1n}, \ldots, \tilde{P}_{nn})^T$ is the operator of the evolution of the matrix $\tilde{P}$ into a vector corresponding to ordering of the elements in the columns.

To solve interval system (4.9), direct and iteration methods, described, for example, in [9, 16], can be used.

Proof. If the conditions of the theorem are satisfied, solutions of Sylvester's "point" equation (4.3) exist, and consequently there is a combined set of solutions (4.4) of interval equation (4.1). Equation (4.1), taking relation (4.6) into account, takes the form

$$\tilde{A}\tilde{P} + \tilde{P}\tilde{F} = \tilde{B}\tilde{K}\tilde{P} \tag{4.10}$$

If the terms $\mathrm{opp}(\tilde{P}\tilde{F})$ and $\mathrm{opp}(\tilde{B}\tilde{K}\tilde{P})$ are added to the left- and right-hand sides of Eq. (4.10), we obtain the equivalent equation

$$\tilde{A}\tilde{P} + \mathrm{opp}(\tilde{B}\tilde{K}\tilde{P}) = \mathrm{opp}(\tilde{P}\tilde{F}) \tag{4.11}$$

where opp is the operation of taking the opposite element in the full interval arithmetics

$$\mathrm{opp}(\tilde{a}) \overset{\mathrm{def}}{=} [-\underline{a}, -\bar{a}], \quad \tilde{a} + \mathrm{opp}(\tilde{a}) = 0$$

By virtue of the property of subdistributivity, the inclusion

$$(\tilde{A} + (-1)(\tilde{B}\tilde{K}))\tilde{P} \subseteq \tilde{A}\tilde{P} + \mathrm{opp}(\tilde{B}\tilde{K}\tilde{P}) = (-1)\tilde{P}\tilde{F}$$

follows from Eq. (4.11).

Let $\tilde{P}$ be a non-degenerate matrix (all point matrices are non-degenerate). Then the latter inclusion is equivalent to the inclusion

$$\tilde{P}^{-1}(\tilde{A} + (-1)(\tilde{B}\tilde{K}))\tilde{P} \subseteq (-1)\tilde{F}$$

which is equivalent to the following assertion: for any $A \in \tilde{A}$ and $B \in \tilde{B}$, the matrix $A_C = (A - BK)$ is similar to any matrix $(-F) \in (-\tilde{F})$, i.e. $\rho(A_C) = \rho(-F)$. Since the similarity occurs for all $A \in \tilde{A}$ and $B \in \tilde{B}$, the inclusion $\rho(\tilde{A}_C) \subseteq \rho(-\tilde{F})$ holds. Consequently, the spectrum of the closed system is a subset of the prescribed spectrum, which it was required to prove.

The procedure proposed for checking the conditions of solvability of the Sylvester-type interval equation enables one to synthesize a controller ensuring robust stability of systems with parameters that are accurately specified up to their lower and upper limits.

5. EXAMPLE

Let $n = 2, x = (x_1, x_2) \in R^2$ and the interval parameters of system (1.1) be such that

$$\tilde{A} = \begin{Vmatrix} [4,5] & 2 \\ 2 & [4,5] \end{Vmatrix}, \quad \tilde{B} = \begin{Vmatrix} [2,4] & 0 \\ 0 & [2,4] \end{Vmatrix} \tag{5.1}$$

The controllability matrix (2.1) for the examined pair $(\tilde{A}, \tilde{B})$ is equal to

$$\tilde{D} = \| \tilde{B} \mid \tilde{A}\tilde{B} \| = \left\Vert \begin{array}{cc|cc} [2,4] & 0 & [8,20] & [4,8] \\ 0 & [2,4] & [4,8] & [8,20] \end{array} \right\Vert$$

In determining the rank of the controllability matrix, we will use a procedure described earlier in [17], For this, we will calculate the symmetrical matrix

$$\tilde{D}\tilde{D}^T = \begin{Vmatrix} [84,480] & [64,320] \\ [64,320] & [84,320] \end{Vmatrix}$$

its median

$$\mathrm{med}(\tilde{D}\tilde{D}^T) = \begin{Vmatrix} 282 & 192 \\ 192 & 202 \end{Vmatrix}$$

the eigenvalues $\lambda_1(\mathrm{med}\tilde{D}\tilde{D}^T) = 90$ and $\lambda_2(\mathrm{med}\tilde{D}\tilde{D}^T) = 474$ and the corresponding eigenvectors

$$\upsilon_1 = \begin{Vmatrix} 0.707 \\ -0.707 \end{Vmatrix}$$

and

$$\upsilon_2 = \begin{Vmatrix} 0.707 \\ 0.707 \end{Vmatrix}$$

Then, from formula (3.7), we find the vector of the discrepancy $\varepsilon_1 = 69.99$ and $\varepsilon_2 = 325.94$, and, using relation (3.6), the eigenvalues of the IM $\tilde{D}\tilde{D}^T$:

$$\tilde{\lambda}_1(\tilde{D}\tilde{D}^T) = [\underline{\lambda}_1,\ \bar{\lambda}_1] = [20.01,\ 159.99],\ \tilde{\lambda}_2(\tilde{D}\tilde{D}^T) = [\underline{\lambda}_2,\ \bar{\lambda}_2] = [148.06,\ 799.94]$$

Then, by relation (3.3), the singular numbers of the controllability matrix are

$$\tilde{\sigma}_1(\tilde{D}) = [\underline{\sigma}_1,\ \bar{\sigma}_1] = [4.47,\ 12.65],\ \tilde{\sigma}_2(\tilde{D}) = [\underline{\sigma}_2,\ \bar{\sigma}_2] = [12.17,\ 28.28]$$

Both singular numbers of the controllability matrix belong to a subset of non-negative intervals and, according to Assertion 3, to rank $\tilde{D} = 2$. Consequently, system (1.1) with the given parameters are controllable, while the controllability margin (3.8) is equal to

$$\mu(\tilde{A},\ \tilde{B}) = \min\{4.47,\ 12.65\} = 4.47$$

We will select

$$\tilde{F} = \begin{Vmatrix} [20,\ 25] & 0 \\ 0 & [20,\ 25] \end{Vmatrix},\quad \tilde{G} = \begin{Vmatrix} [12,\ 15] & 1 \\ 1 & [12,\ 15] \end{Vmatrix}$$

Solving the Sylvester equation (4.1), we find

$$\tilde{P} = \begin{Vmatrix} [1,\ 2] & 0 \\ 0 & [1,\ 2] \end{Vmatrix}$$

and from Eq. (4.6) we determine the parameters of the controller (4.5):

$$\tilde{K} = \begin{Vmatrix} [12,\ 7.5] & [1,\ 0.5] \\ [1,\ 0.5] & [12,\ 7.5] \end{Vmatrix}$$

Then, system (1.1), closed by the given controller, takes the form

$$\dot{x} = \tilde{A}_C x = (\tilde{A} - \tilde{B}\tilde{K})x = \begin{Vmatrix} [-20,\ -25] & [0] \\ [0] & [-20,\ -25] \end{Vmatrix} x$$

The eigenvalues of the closed-contour matrix which are equal to

$$\tilde{\lambda}_{1,2}(A_C) = [\underline{\lambda},\ \bar{\lambda}] = [-20,\ -25]$$

are positioned in the left-hand half-plane for all values of the parameters (5.1) from the prescribed intervals, which confirms the robust stability of the system with the controller synthesized.

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